Single Lens Lightcurve Modeling

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Outline.

- Basic Equations.
- Limits.
- Derivatives.
- Degeneracies.
- Fitting.
  - Linear Fits.
  - Multiple Observatories.
  - Nonlinear fits.
- Complications.
- Summary & References.

Simplest Model.

- Single Source
- Single Lens.
- Rectilinear Trajectory.
  - No acceleration in lens, source, observer.
- Point source.
- Cospatial observers.

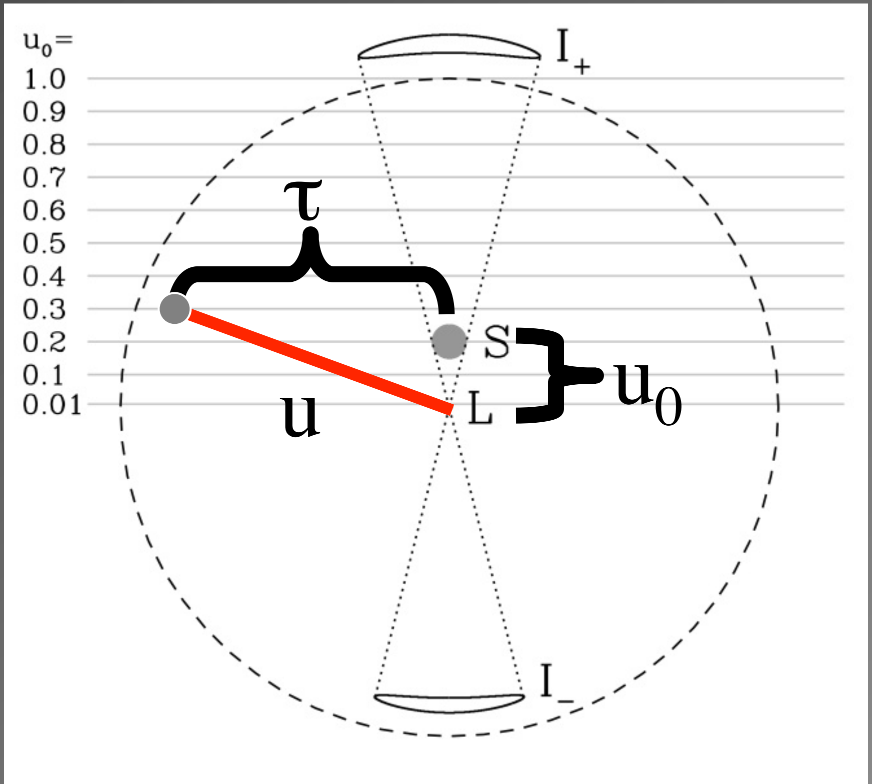
Rectilinear Trajectories.

- Rectilinear trajectory

$$u(t) = \left(\tau^2 + u_0^2\right)^{1/2}$$

(u, τ, u_0 in units of θ_E)

where $$\tau = \frac{t - t_0}{t_E}$$



Parameters: t_0, t_E, u_0

Einstein Timescale.

- Time to cross the angular Einstein ring radius.

$$t_E \equiv \frac{\theta_E}{\mu}$$

- μ is the relative lens-source proper motion.
- Timescale is a degenerate combination of the lens mass, lens and source distances, and the lens and source relative proper motion.
- Median timescale for events toward the bulge is ~20 days; range from a few to hundreds of days.

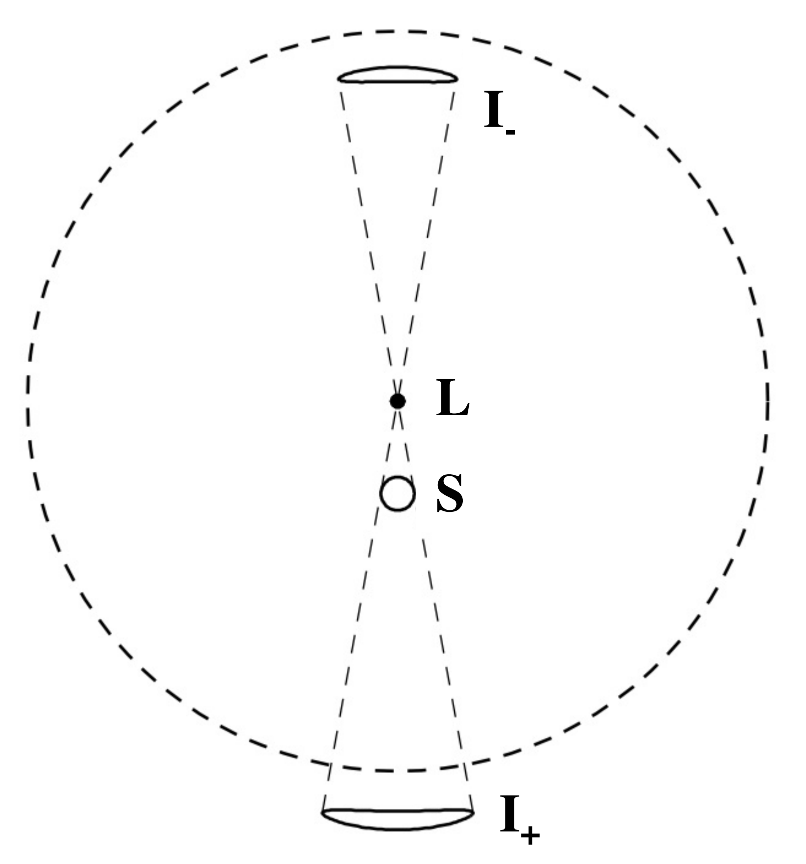
Magnifications.

- Can be derived simply:

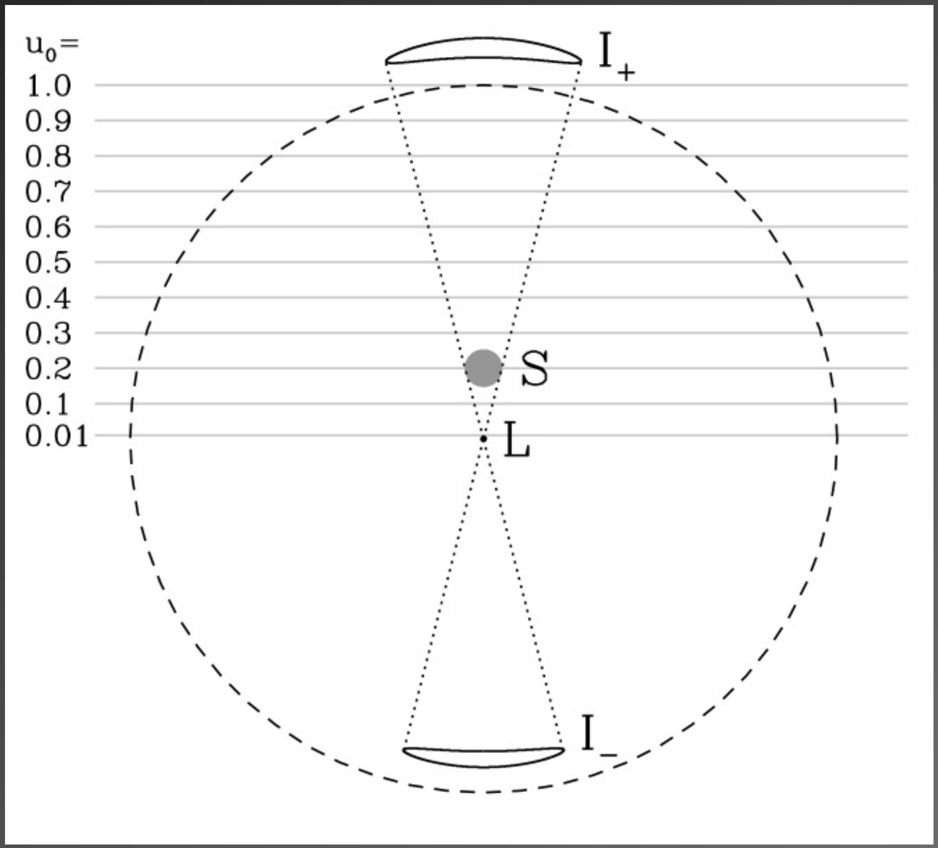
$$A_{\pm} = \left| \frac{y_{\pm}}{u} \frac{dy_{\pm}}{du} \right| = \frac{1}{2} \left(\frac{u^2 + 2}{u\sqrt{u^2 + 4}} \pm 1 \right)$$

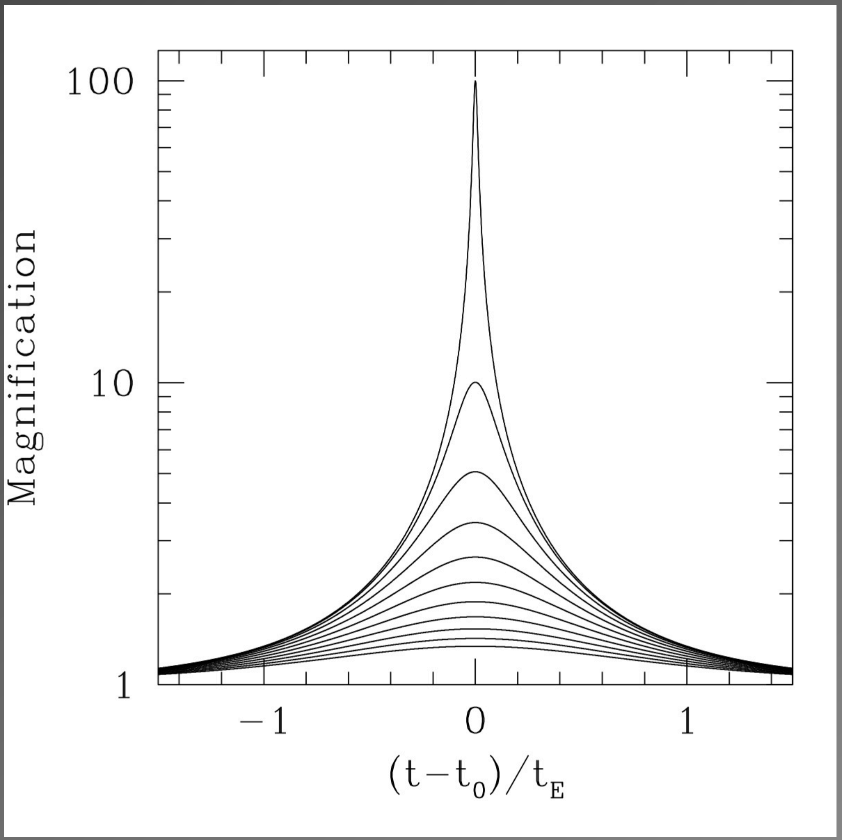
- Note that: $A_{+} - A_{-} = 1$
- Total magnification:

$$A = \frac{u^2 + 2}{u\sqrt{u^2 + 4}}$$



Magnification vs. Time





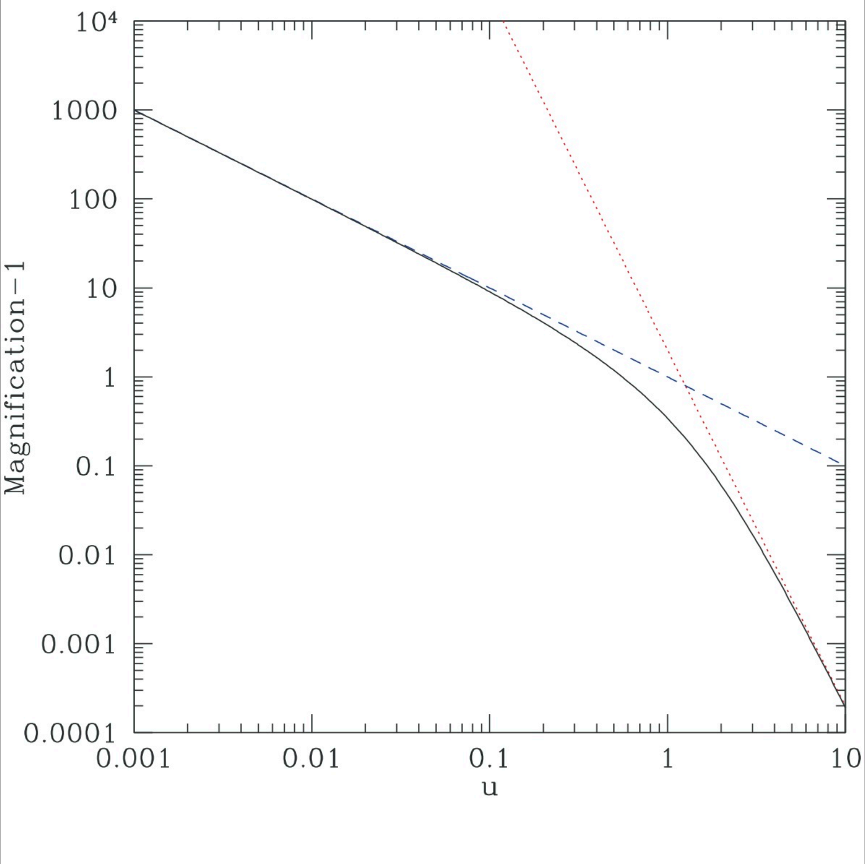
- Three parameter family of curves.
- Parameters: t_0, t_E, u_0

Limits.

- Limits for low and high mag.events.

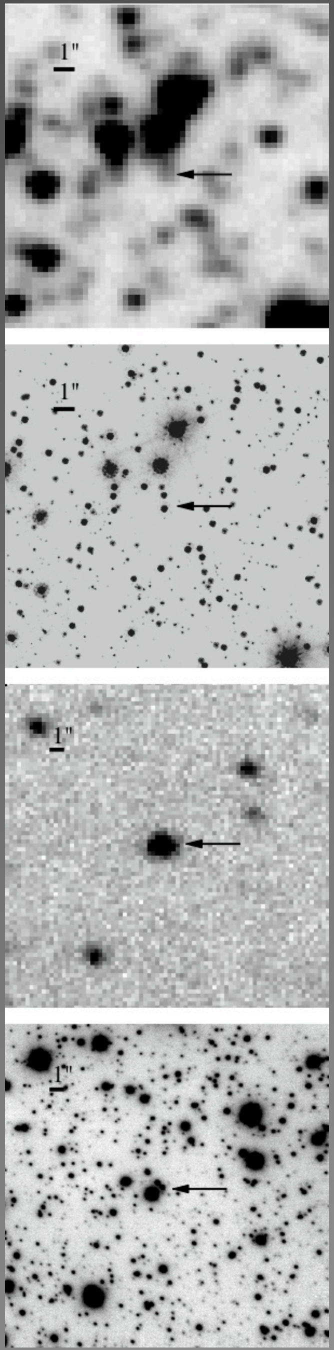
$$A \approx 1 + 2u^{-4} \text{ for } u >> 1$$

$$A \approx u^{-1} \text{ for } u << 1$$



Flux (not magnification).

- Magnification is ***not*** directly observable.
- We observe the flux from the lensed source *and* any unresolved “blends”.
- Includes light from:
  - Lens.
  - Companions to the lens.
  - Companions to the source.
  - Unrelated stars.



Simplest form.

$$F(t) = F_s A[u(t)] + F_l + \sum F_{cs,i} + \sum F_{cl,i} + \sum F_{b,i}$$

$$F_b$$

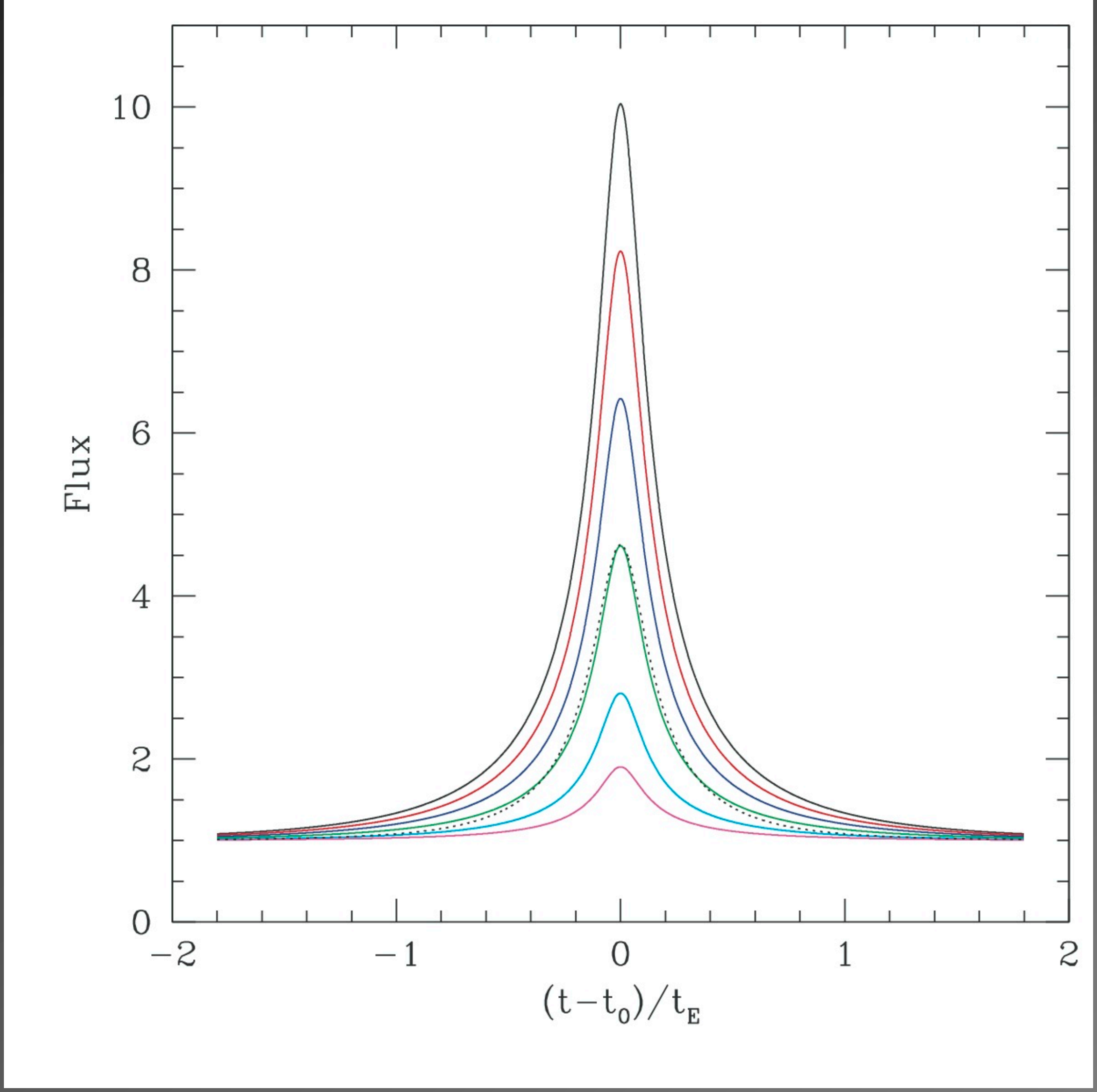
$$F(t) = F_s A[u(t)] + F_b$$

Single lens model.

$$F(t) = F_s A[u(t; t_0, t_E, u_0)] + F_b$$

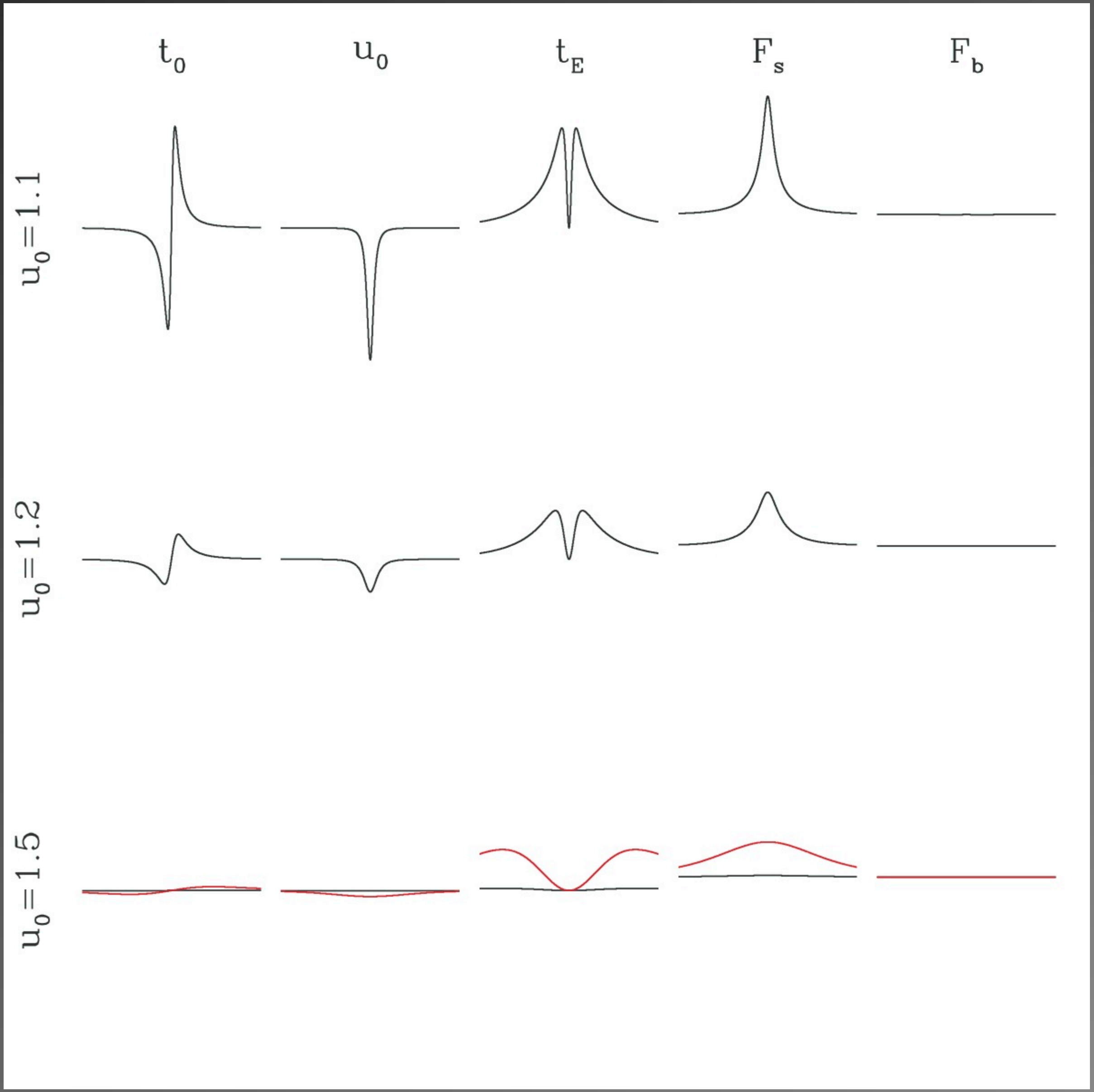
- Five parameters.
  - t_0, t_E, u_0, F_S, F_B
- Note that the flux depends:
  - Linearly on F_S, F_B
  - Non-linearly on t_0, t_E, u_0
- There are four basic observables:
  - Baseline flux = $F_S + F_B$
  - Peak Flux.
  - Time of peak flux = t_0
  - Duration (i.e., full width half maximum)

Blended Light Curves.



- $f = F_S/(F_S + F_B) = 1.0$
- $f = 0.8$
- $f = 0.6$
- $f = 0.4$
- $f = 0.2$
- $f = 0.1$

Derivatives.



Degeneracies - General.

- Five parameters.
  - t_0, t_E, u_0, F_S, F_B
- Four basic observables:
  - Baseline flux, Peak Flux, Time of peak flux, FWHM

$$\frac{F(t)}{F_s + F_b} = fA[u(t;t_0,t_E,u_0)] + (1-f)$$

- Four parameters:
  - t_0, t_E, u_0, f
- Three observables:
  - Peak Flux, Time of peak flux, FWHM

Degeneracies - Low Mag.

- In the limit of $u_0>>1$, perfect degeneracy:

$$\frac{F(t)}{F_s + F_b} \approx f[1+2u^{-4}] + (1-f) = 1 + 2fu^{-4}$$

- Observed flux is invariant under the substitution:

$$f' = fC^4;\ u_0' = u_0 C;\ t_E' = t_E C^{-1}$$

Degeneracies - High Mag.

- In the limit of $u<<1$, perfect degeneracy:

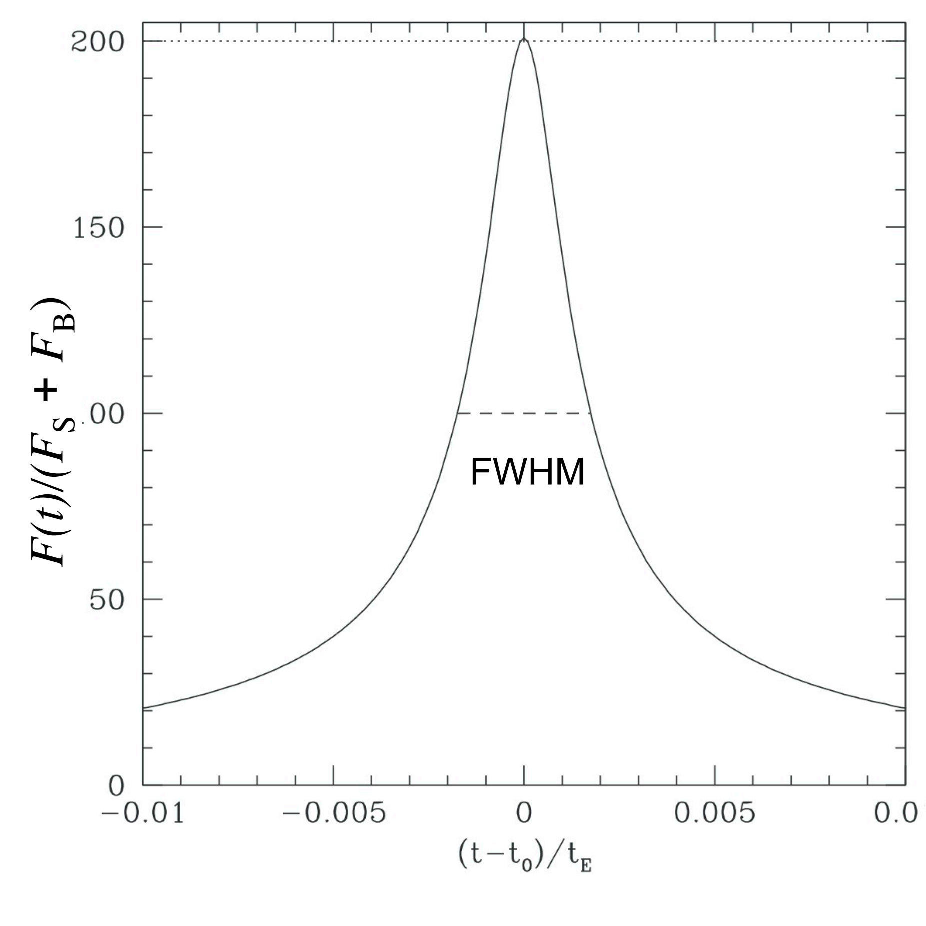
$$\frac{F(t)}{F_s + F_b} \approx f[1+u^{-1}] + (1-f) = 1 + fu^{-1}$$

- Observed flux is invariant under the substitution:

$$f' = fC;\; u_0' = u_0 C;\; t_E' = t_E C^{-1}$$

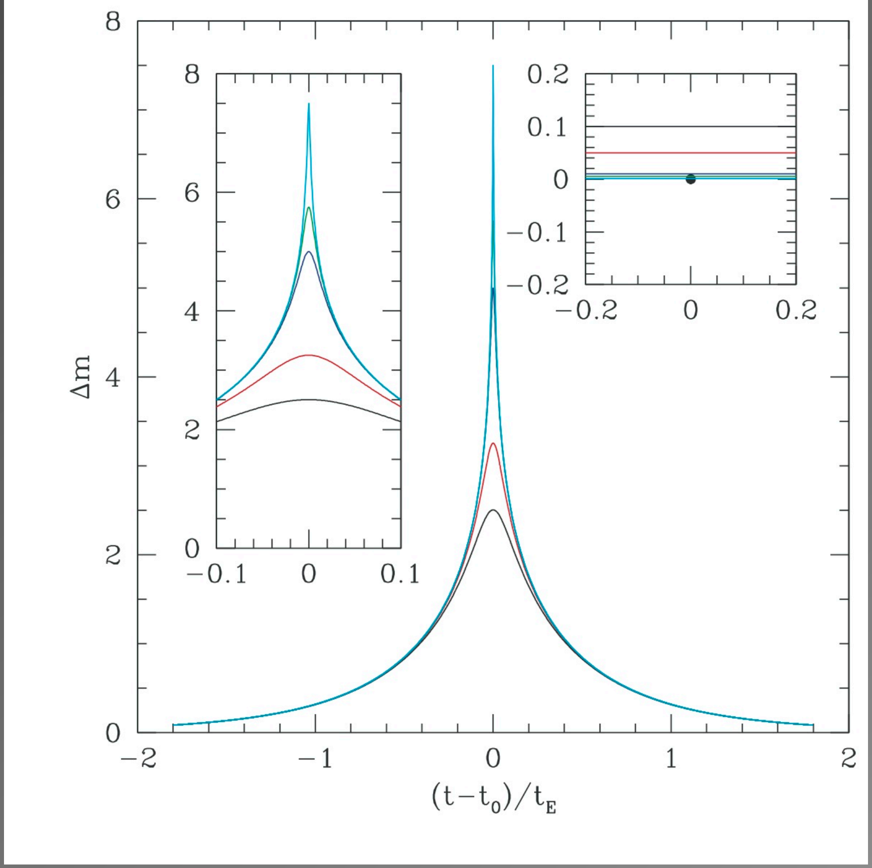
Degeneracies - High Mag.

- Four parameters:
  - t_0, t_E, u_0, f
- Three observables:
  - Peak Flux = F_S / u_0
  - Time of peak flux = t_0
  - FWHM = $12^{-1/2} u_0 t_E$



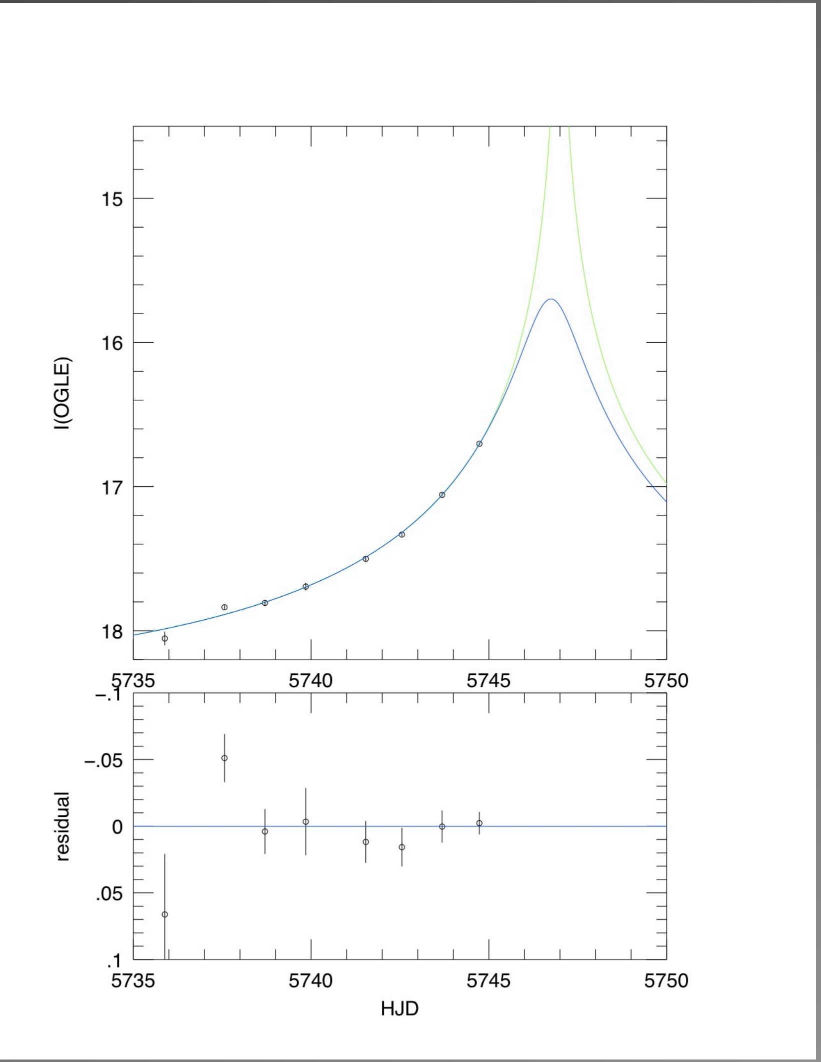
Before peak.

- High magnification light curves appear very similar until just before peak.



Before peak.

- Degeneracy with t_0 for data pre-peak.
- Higher magnification fits generally occur later.

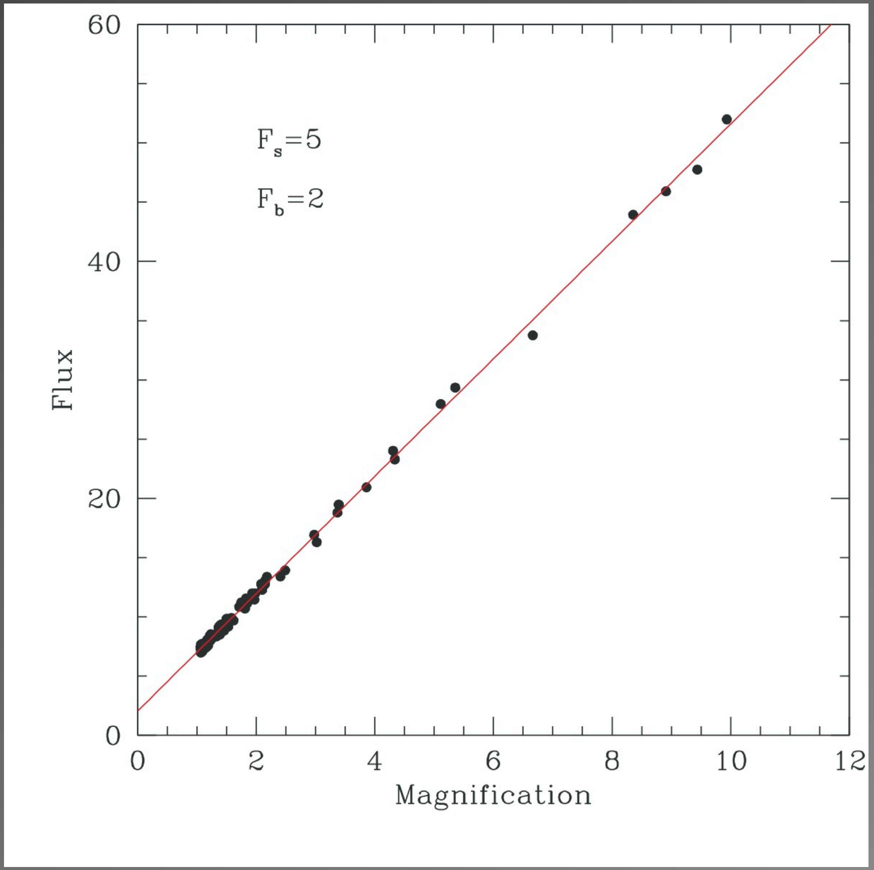


Fitting.

- Basic Problem.
- Data: $F_k,\ \sigma_k$, taken at times t_k
- Model: $F(t) = F_s A[u(t; t_0, t_E, u_0)] + F_b$
- Parameters: $\vec{a} = [t_0, t_E, u_0, F_s, F_b]$
- Want to maximize the likelihood wrt the parameters: $L = \exp(-\chi^2/2)$ **(Gaussian errors)**
- Where: $$\chi^2 = \sum_k \left(\frac{F_k - F(t_k)}{\sigma_k} \right)^2$$ **(uncorrelated errors)**

Linear Fits.

- F_S and F_B are linear parameters.
- Given a set of values of t_0, t_E, u_0 [and so $A(t)$], can fit for F_S and F_B analytically.



Linear Fits.

Steps:

1. Form the covariance matrix:

$$C_{ij} = b_{ij}^{-1}, \quad b_{ij} = \sum_k \left.\frac{\partial F(t)}{\partial a_i}\right|_{t=t_k} \left.\frac{\partial F(t)}{\partial a_j}\right|_{t=t_k} \sigma_k^{-2}$$

2. And the vector:

$$d_i = \sum_k F(t_k) \left.\frac{\partial F(t)}{\partial a_i}\right|_{t=t_k} \sigma_k^{-2}$$

3. The parameters which minimize χ^2 are then:

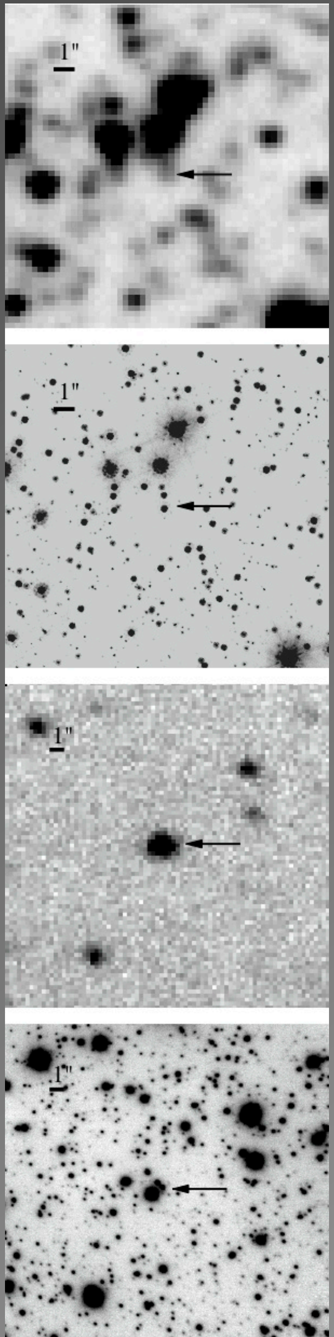
$$a_{\text{best},i} = \sum_j C_{ij} d_j$$

Mutliple Observatories and/or Filters.

- Generally, F_S and F_B will depend on the filter and observatory.
- Even assuming the same wavelength response, blend flux can change for different observatories.
- In reality, wavelength responses will vary.

$$F_1(t) = F_{s,1}A[u(t;t_0,t_E,u_0)] + F_{b,2}$$
$$F_2(t) = F_{s,2}A[u(t;t_0,t_E,u_0)] + F_{b,2}$$
$$\ldots$$

- Total number of parameters = $3+2\times N_o$
- Incur no additional “expense” because they are linear.



Non-linear minimization.

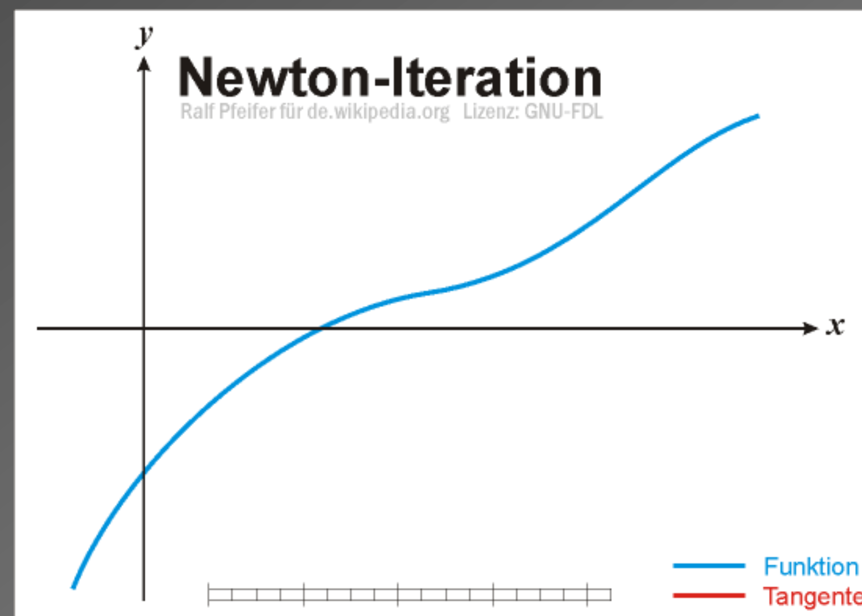
- t_0, t_E, u_0 are non-linearly related to *F(t).*
- The general problem of finding non-linear parameters which minimize χ^2 (the global "best-fit") is hard.
  - False (local) minima.
  - Poorly-behaved likelihood surfaces.
  - Strong continuous degeneracies.
  - Discrete degeneracies.
- Fortunately, the single-lens problem is not too problematic (for good sampling).

Methods.

- Grid searches.
  - Inefficient.
- Newton’s Method.
- Markov Chain Monte Carlo.
  - Not really designed for minimization.
- Canned routines:
  - AMOEBA (Numerical Recipes)
  - MPFIT (IDL)
- Minimization can be made faster and more robust by stepping in parameters that are more directly related to the data.
  - For example, for high-magnification events: F_{max}, *FWHM*

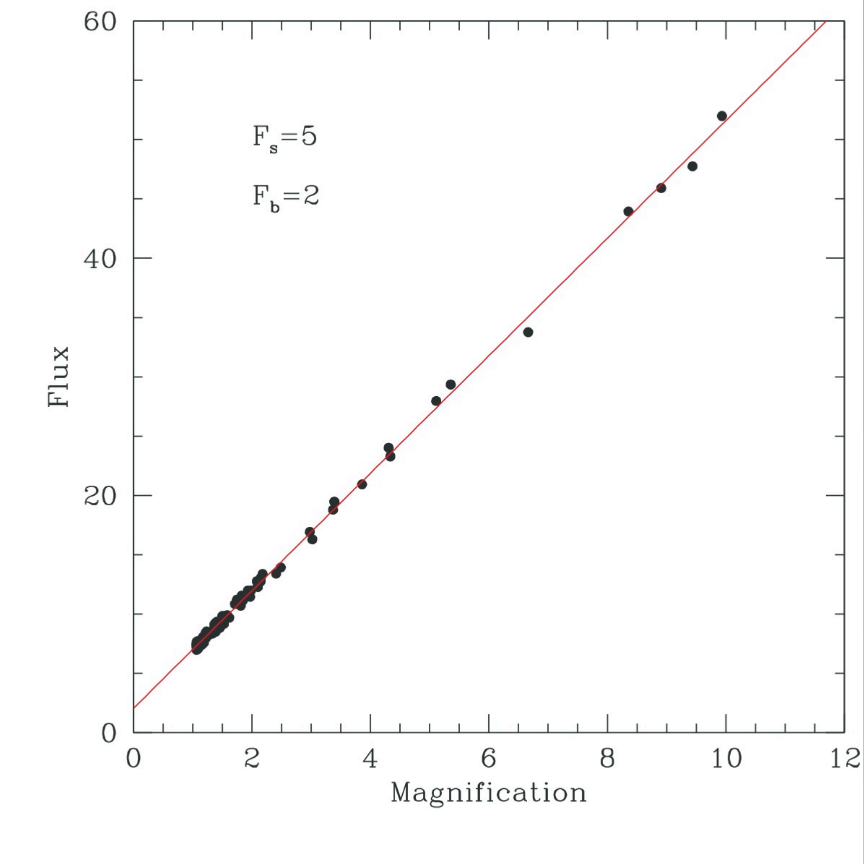
Newton’s Method.

- Find the root of a function *f(x)* .
- Begin with a guess for the parameter x_0.
- Evaluate: $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$
- Iterate: $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$
- Can be extended to *N* dimensions.
- Basis of `sfit` program.



Hybrid Fitting.

- All fits to microlensing light curves involve a hybrid method:
  1. Start with a trial set of non-linear parameters, which specify the magnification versus time.
  2. Linearly fit the blend and source fluxes for that trial set.
  3. Evaluate χ^2 for that set of non-linear parameters.
  4. Minimize χ^2 .
- Note that the uncertainties evaluated based on this χ^2 are underestimated: do not account for the uncertainty in F_s, F_b at fixed magnification.



Complications.

- Correlated uncertainties.
- Poor sampling and incomplete coverage.
  - May require fixing parameters.
- Higher-order effects.
  - Parallax and xallarap.
  - Finite source.
- Most methods fail (miserably) for most binary lenses.

Summary.

- Simplest microlensing light curve is a function of $3+2\times N_o$ parameters.
  - Three non-linear parameters t_0, t_E, u_0
  - $2\times N_o$ linear parameters F_s, F_b for each observatory/filter combination.
- Four basic observables: t_E, u_0, F_s, F_b can be degenerate.

- Linear parameters can be found analytically.